Europ. J. Combinatorics (1981) **2**, 299–303

New Combinatorial Characterizations of Generalized Quadrangles

J. A. Thas

1. Introduction

Let $\mathcal{S}=(\mathcal{P},\mathcal{B},I)$ be a generalized quadrangle of order (s,t). For a fixed point p, define the following condition: $(M)_p$ *For any two lines L, M of $\mathcal{S}$ incident with p, the group of collineations of $\mathcal{S}$ fixing L and M pointwise and p linewise is transitive on the lines $(\neq L)$ incident with a given point x on L $(x\neq p)$.*

Then $\mathcal{S}$ is said to satisfy condition (M) provided it satisfies $(M)_p$ for all points $p\in\mathcal{P}$. Let $(\hat{M})$ be the dual of (M). If $\mathcal{S}$ satisfies both (M) and $(\hat{M})$ and if $s\neq 1\neq t$, it is called a *Moufang* generalized quadrangle [12], and by a celebrated theorem of Tits [11, 13] *a thick generalized quadrangle is a Moufang generalized quadrangle iff it is one of the classical examples or their duals.* Recently Tits proved that the theorem is also valid in the infinite case [13, 14].

Another characterization of all classical generalized quadrangles was given by Buekenhout and Lefèvre [2]: *The generalized quadrangle $\mathcal{S}$ of order (s,t) is classical iff it is embeddable in a projective space $PG(n,s)$.* In the infinite case the theorem was proved by K. J. Dienst [4].

Here we introduce a new condition (A), and its dual $(\hat{A})$, for generalized quadrangles (Section 3), and we show (Section 4) *that a thick generalized quadrangle is a classical or a dual classical generalized quadrangle iff it satisfies one of the conditions (A) or $(\hat{A})$.* Finally, in Section 5 a new characterization of the dual of $T(\mathcal{O})$, the Tits' quadrangle arising from an ovoid $\mathcal{O}$, is given.

2. Basic Concepts and Results

A generalized quadrangle of order (s,t) $(s\geqslant 1, t\geqslant 1)$ is an incidence structure $\mathcal{S}=(\mathcal{P},\mathcal{B},I)$ with pointset $\mathcal{P}$, lineset $\mathcal{B}$, and symmetric point–line incidence relation I satisfying the following axioms:

(i) *Each point (resp. line) is incident with $1+t$ lines (resp. $1+s$ points).*
(ii) *Two points are incident with at most one line.*
(iii) *If x is a point not incident with a line L, there is a unique point-line pair $(y,M)\in\mathcal{P}\times\mathcal{B}$ with $xIMIyIL$.*

If $s\neq 1\neq t$, $\mathcal{S}$ is called *thick*. When the points x, y (resp. lines L, M) of $\mathcal{S}$ are collinear (resp. concurrent), we write $x\sim y$ (resp. $L\sim M$); otherwise $x\not\sim y$ (resp. $L\not\sim M$). For $(x,L)\in\mathcal{P}\times\mathcal{B}$ put $x^\perp=\{y\in\mathcal{P}\,\|\,y\sim x\}$ and $L^\perp=\{M\in\mathcal{B}\,\|\,L\sim M\}$. The *trace* of a pair of distinct points (x,y) is defined to be the set $x^\perp\cap y^\perp$ and is denoted $\{x,y\}^\perp$. More generally, if $A\subset\mathcal{P}$, A "perp." is defined by $A^\perp=\cap\{x^\perp\,\|\,x\in A\}$. For $x\neq y$, the *span* of the pair (x,y) is $\{x,y\}^{\perp\perp}=\{u\in\mathcal{P}\,\|\,u\in z^\perp$ for all $z\in x^\perp\cap y^\perp\}$. And the *closure* of the pair (x,y) is $\mathrm{cl}\,(x,y)=\{z\in\mathcal{P}\,\|\,z^\perp\cap\{x,y\}^{\perp\perp}\neq\varnothing\}$. We say that the pair (x,y) is *regular* if either $x\sim y$ or $x\not\sim y$ and $|\{x,y\}^{\perp\perp}|=t+1$. The point x is *regular* provided (x,y) is regular for all $y\in\mathcal{P}$, $y\neq x$. If $\mathcal{S}$ contains a regular pair of non-collinear points, then $s=1$ or $s\geqslant t$. A *triad* (*of points*) is a triple of pairwise non-collinear points.

There is a point–line duality for generalized quadrangles (of order (s,t)) which interchanges "point" and "line", interchanges s and t, in any definition or theorem. Normally,

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we assume without further notice that the dual of a given theorem or definition is also given. A line L is *coregular* provided each point incident with L is regular.

The generalized quadrangle arising from a non-singular hyperquadric Q of index 2 in $PG(d, q)$, $d = 3$, 4 or 5, is denoted by $Q(d, q)$; the generalized quadrangle arising from a non-singular Hermitian variety H in $PG(d, q^2)$, $d = 3$ or 4, is denoted by $H(d, q^2)$; the generalized quadrangle formed by the points of $PG(3, q)$ together with all totally isotropic lines with respect to a symplectic polarity of $PG(3, q)$ is denoted by $W(3, q)$. These are the so-called *classical* generalized quadrangles. We remark that $W(3, q)$ is isomorphic to the dual of $Q(4, q)$, that $H(3, q^2)$ is isomorphic to the dual of $Q(5, q)$, and that all classical generalized quadrangles but $Q(3, q)$ are thick [10].

The Tits' quadrangle arising from an ovoid $\mathcal{O}$ of $PG(3, q)$ [3] will be denoted by $T(\mathcal{O})$. It is isomorphic to $Q(5, q)$ iff $\mathcal{O}$ is an elliptic quadric [10].

The generalized quadrangle $\mathcal{S}$ is said to satisfy *property* (H) provided $x \in \mathrm{cl}(y, z)$ implies $y \in \mathrm{cl}(x, z)$ for any triad (x, y, z). Also we introduce the following *axiom* $(\hat{D})$: If l_1, l_2, m_1, m_2, u_1, u_2, n_1 are distinct points for which $l_1 \not\sim m_2$, $l_1 \sim l_2$, $m_1 \sim m_2$, $l_1 \sim u_1 \sim m_1$, $l_2 \sim u_2 \sim m_2$, $u_1 \sim u_2$, $l_1 \sim n_1 \sim m_1$ and for which l_1l_2, m_1m_2, l_1u_1, u_1m_1, l_2u_2, u_2m_2, u_1u_2, l_1n_1, n_1m_1 are distinct lines, then there exists a point n_2 with $l_2 \sim n_2 \sim m_2$ and $n_1 \sim n_2$. Theorem (a) and the dual of (b) are proved in [10, 8] and [9] respectively.

(a) *If $\mathcal{S}$ satisfies property (H), then one of the following must occur:*
  (i) *each point of $\mathcal{S}$ is regular (implying $s = 1$ or $s \geq t$),*
  (ii) *$|\{x, y\}^{\perp\perp}| = 2$ for all $x, y \in \mathcal{P}$ with $x \not\sim y$, or*
  (iii) *$\mathcal{S}$ is isomorphic to $H(4, s)$.*

(b) *If $2 < t < s$ and $\mathcal{S}$ contains a coregular line L such that $(\hat{D})$ is satisfied whenever l_1 or l_2 is on L, then $s = t^2$ and $\mathcal{S}$ is isomorphic to the dual of a $T(\mathcal{O})$ of Tits.*
  If $\mathcal{S}$ is a generalized quadrangle with parameters s, t, where $2 < t < s$, for which each point is regular and for which $(\hat{D})$ is satisfied, then $t^2 = s$ and $\mathcal{S}$ is isomorphic to $H(3, s)$.

From (b) there easily follows

(c) *If $2 < t < s$ and $\mathcal{S}$ contains a coregular line L such that $(\hat{D})$ is satisfied whenever l_1 (resp. l_2) is on L, then $s = t^2$ and $\mathcal{S}$ is isomorphic to the dual of a $T(\mathcal{O})$ of Tits.*

PROOF OF (c). Suppose that $(\hat{D})$ is satisfied whenever l_1 is on the coregular line L. Then we prove that $(\hat{D})$ holds whenever l_2 is on L. Since l_2 is on L, the point l_2 is regular. If $l_2 \sim m_1$, then $u_2, m_1, l_1 \in \{l_2, u_1\}^{\perp}$, and so $m_2 \sim l_1$, a contradiction. So we always have $l_2 \not\sim m_1$. Let v_2 be a point for which $v_2 \sim l_2$, $v_2 \sim m_2$ and $v_2 \neq u_2$. Then, by axiom $(\hat{D})$, there is a point v_1 such that $v_1 \sim l_1$, $v_1 \sim m_1$, $v_1 \sim v_2$. Since $l_2 \not\sim m_1$ we have $v_1 \neq u_1$. From $l_2 \not\sim m_1$ it also follows that the point v_1 is uniquely defined by v_2, and that $v_2 \mapsto v_1$ defines an injection from $\{l_2, m_2\}^{\perp} - \{u_2\}$ into $\{l_1, m_1\}^{\perp} - \{u_1\}$. Since $|\{l_2, m_2\}^{\perp} - \{u_2\}| = |\{l_1, m_1\}^{\perp} - \{u_1\}| = t$, the injection $v_2 \mapsto v_1$ is a bijection from $\{l_2, m_2\}^{\perp} - \{u_2\}$ onto $\{l_1, m_1\}^{\perp} - \{u_1\}$. It immediately follows that $(\hat{D})$ is satisfied whenever l_2 is on L. And so, the first part of Theorem (b) applies.

The other case is completely analogous.

3. PROPERTIES (A) AND $(\hat{A})$

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a generalized quadrangle of order (s, t). If $\mathcal{B}^{\perp\perp} = \{\{x, y\}^{\perp\perp}, x \not\sim y\}$, then let $\mathcal{S}^{\perp\perp} = (\mathcal{P}, \mathcal{B}^{\perp\perp}, \in)$. For $x \in \mathcal{P}$ we say $\mathcal{S}$ satisfies *property* $(A)_x$ if for any $M = \{y, z\}^{\perp\perp} \in \mathcal{B}^{\perp\perp}$ with $x \in \{y, z\}^{\perp}$, and any $u \in \mathrm{cl}(y, z) \cap (x^{\perp} - \{x\})$ (or $u \neq x$ and $uIxk$ for some $k \in M$) with $u \notin M$, the substructure of $\mathcal{S}^{\perp\perp}$ generated by M and u is a dual affine plane. The generalized quadrangle $\mathcal{S}$ is said to satisfy *property* (A) if it satisfies $(A)_x$ for all $x \in \mathscr{P}$. The duals of these properties are denoted resp. $(\hat{A})_L$ and $(\hat{A})$. We notice that property (A) is equivalent to the following: if $M = \{y, z\}^{\perp\perp} \in \mathscr{B}^{\perp\perp}$ and $u \in \mathrm{cl}(y, z)$, $u \notin M \cup M^{\perp}$, then u and M generate a dual affine plane in $\mathscr{S}^{\perp\perp}$.

4. Theorem

The thick generalized quadrangle $\mathscr{S}$ is a classical or a dual classical generalized quadrangle iff it satisfies one of the conditions (A) or $(\hat{A})$.

Proof. It is not difficult (and an interesting exercise) to check that a thick classical or dual classical generalized quadrangle satisfies one of the conditions (A) or $(\hat{A})$.

Next let us assume that the thick generalized quadrangle $\mathscr{S} = (\mathscr{P}, \mathscr{B}, I)$ satisfies (A). We shall prove that property (H) is satisfied. So we consider a triad (u, y, z) for which $u \in \mathrm{cl}(y, z)$. Let π be the dual affine plane generated by $\{y, z\}^{\perp\perp}$ and u in $\mathscr{S}^{\perp\perp}$. Evidently $\{z, u\}^{\perp\perp}$ is a line of π. In π the point y is not collinear with exactly one point of $\{z, u\}^{\perp\perp}$, i.e., in $\mathscr{S}$ the point y is collinear with exactly one point of $\{z, u\}^{\perp\perp}$. Hence $y \in \mathrm{cl}(z, u)$, and so (H) is satisfied.

Now we assume that $\mathscr{S} \not\cong H(4, s)$. If $|\{y, z\}^{\perp\perp}| = 2$ for all $y, z \in \mathscr{P}$ with $y \not\sim z$, then for any $M = \{y, z\}^{\perp\perp} \in \mathscr{B}^{\perp\perp}$ and any $u \in \mathrm{cl}(y, z) - \{y, z\}^{\perp}$ with $u \notin M$ (such a u exists by the thickness of $\mathscr{S}$), the substructure of $\mathscr{S}^{\perp\perp}$ generated by M and u has three points and consequently is not a dual affine plane, a contradiction. Then, from Section 2, it follows that every point of $\mathscr{S}$ is regular. Since $\mathscr{S}$ is thick we have $t \neq 1$ and $s \geq t$. If $s = t$, then it is well known that $\mathscr{S}$ is isomorphic to $W(3, s)$ [10]. If $t = 2 < s$, then $\mathscr{S}$ is isomorphic to $H(3, 4)$ [5, 6, 7]. Hence we now suppose that $2 < t < s$.

We shall prove that $\mathscr{S}$ satisfies property $(\hat{D})$. Consider points l_1, l_2, m_1, m_2, u_1, u_2, n_1 as in axiom $(\hat{D})$. Since the pair (l_2, m_2) is regular the point u_1 is element of $\mathrm{cl}(l_2, m_2)$. We consider the dual affine plane π generated by $\{l_2, m_2\}^{\perp\perp}$ and u_1 in $\mathscr{S}^{\perp\perp}$. Lines of π are $\{l_2, u_1\}^{\perp\perp}$ and $\{m_2, u_1\}^{\perp\perp}$. Since l_1 (resp. m_1) is an element of $\{u_1, l_2\}^{\perp}$ (resp. $\{u_1, m_2\}^{\perp}$), the line $l_1 n_1$ (resp. $m_1 n_1$) contains a point r (resp. r') of $\{u_1, l_2\}^{\perp\perp}$ (resp. $\{u_1, m_2\}^{\perp\perp}$). Then $R = \{r, r'\}^{\perp\perp}$ is a line of π. As any two lines of π intersect, the lines $\{l_2, m_2\}^{\perp\perp}$ and R of π have a point r'' in common. We have $r'' \sim n_1$. If n_2 is the point of $n_1 r''$ which is collinear (in $\mathscr{S}$) with l_2, then $m_2 \in \{l_2, r''\}^{\perp\perp}$ and $n_2 \in \{l_2, r''\}^{\perp}$ imply $n_2 \sim m_2$. Hence $n_2 \sim n_1$, $n_2 \sim l_2$, and $n_2 \sim m_2$. Consequently $(\hat{D})$ is satisfied, and so $\mathscr{S}$ is isomorphic to $H(3, s)$.

We have proved that if the thick quadrangle $\mathscr{S}$ satisfies (A), then $\mathscr{S}$ is isomorphic to $H(4, s)$, $W(3, s)$, or $H(3, s)$. Hence, if the thick quadrangle $\mathscr{S}$ satisfies one of the conditions (A) or $(\hat{A})$, then it is isomorphic to $H(4, s)$, the dual of $H(4, t)$, $W(3, s)$, $Q(4, s)$, $H(3, s)$ or $Q(5, s)$.

5. Theorem

Let $\mathscr{S}$ be a thick generalized quadrangle with $s \neq t$. Then $\mathscr{S}$ is isomorphic to the dual of a Tits' quadrangle $T(\mathscr{O})$, arising from an ovoid $\mathscr{O}$, iff $(A)_x$ is satisfied for all points x incident with some coregular line L.

Proof. Let $\mathscr{S}$ be the dual of the Tits' quadrangle $T(\mathscr{O})$ arising from the ovoid $\mathscr{O}$ of $PG(3, q)$. Then property $(A)_x$ is satisfied for every point x incident with the (coregular) line of type (iii) [3].

Now let $\mathscr{S}$ be a thick generalized quadrangle with $s \neq t$ such that $(A)_x$ is satisfied for every point x on some coregular line L, Since $\mathscr{S}$ contains regular points and $1 < s \neq t$, we have $s > t > 1$. We shall prove that $(\hat{D})$ is satisfied whenever l_1 is on L. Hence consider points l_1, l_2, m_1, m_2, u_1, u_2, n_1 as in axiom $(\hat{D})$. Since the pair (l_2, u_1) is regular the point n_1 is an element of $\mathrm{cl}(l_2, u_1)$. We consider the dual affine plane π generated by $\{l_2, u_1\}^{\perp\perp}$ and n_1 in $\mathcal{S}^{\perp\perp}$. Lines of π are $\{u_1, n_1\}^{\perp\perp}$ and $\{l_2, n_1\}^{\perp\perp}$. Since u_2 (resp. m_1) is an element of $\{l_2, u_1\}^{\perp}$ (resp. $\{u_1, n_1\}^{\perp}$), the line u_2m_2 (resp. m_1m_2) contains a point r (resp. r') of $\{l_2, u_1\}^{\perp\perp}$ (resp. $\{u_1, n_1\}^{\perp\perp}$). Then $R = \{r, r'\}^{\perp\perp}$ is a line of π. As any two lines of π intersect, the lines $\{l_2, n_1\}^{\perp\perp}$ and R of π have a point r'' in common. We have $r'' \sim m_2$. If n_2 is the point of m_2r'' which is collinear (in $\mathcal{S}$) with l_2, then from $n_1 \in \{l_2, r''\}^{\perp\perp}$ and $n_2 \in \{l_2, r''\}^{\perp}$ it follows that $n_2 \sim n_1$. Hence $n_2 \sim n_1$, $n_2 \sim l_2$, and $n_2 \sim m_2$. Consequently $\hat{D}$) is satisfied. If $t \neq 2$, Theorem (c) in Section 2 tells us that $s = t^2$ and that $\mathcal{S}$ is isomorphic to the dual of a Tits' quadrangle $T(\mathcal{O})$ arising from an ovoid $\mathcal{O}$. If $t = 2$, then it is well known that $\mathcal{S}$ is isomorphic to $H(3, 4)$, the unique generalized quadrangle of order (4.2) [5, 6, 7].

Corollaries

(a) *Let $\mathcal{S}$ be a thick generalized quadrangle with $s \neq t$ and s odd. Then $\mathcal{S}$ is isomorphic to $H(3, s)$ iff $(A)_x$ is satisfied for all points x incident with some coregular line L.*

Proof. If $\mathcal{S} \cong H(3, s)$, then $(A)_x$ is satisfied for every point x of $\mathcal{S}$. Conversely, if $(A)_x$ is satisfied for all points x incident with some coregular line L, then $\mathcal{S}$ is isomorphic to the dual of $T(\mathcal{O})$ for some ovoid $\mathcal{O}$. Since s is odd, $\mathcal{O}$ is an elliptic quadric [1], and so $T(\mathcal{O}) \cong Q(5, s)$. Hence $\mathcal{S} \cong H(3, s)$.

(b) *Let $\mathcal{S}$ be a thick generalized quadrangle with $s \neq t$ and all points regular. Then $\mathcal{S}$ is isomorphic to $H(3, s)$ iff $(A)_x$ is satisfied for all points x incident with some line L.*

Proof. If $\mathcal{S} \cong H(3, s)$, then $(A)_x$ is satisfied for every point x of $\mathcal{S}$. Conversely, if $(A)_x$ is satisfied for all points x incident with some coregular line L, then $\mathcal{S}$ is isomorphic to the dual of $T(\mathcal{O})$ for some ovoid $\mathcal{O}$. Since all points of $\mathcal{S}$ are regular, all lines of $T(\mathcal{O})$ are regular, and so all ovals on $\mathcal{O}$ are conics. Consequently $\mathcal{O}$ is an elliptic quadratic [1]. Hence $T(\mathcal{O}) \cong Q(5, s)$ and so $\mathcal{S} \cong H(3, s)$.

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Received 1 August 1980 and in revised form 6 April 1981

J. A. THAS
Seminar of Geometry and Combinatorics, University of Ghent,
Krijgslaan 271, 9000 Gent, Belgium